The Gaussian Copula

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September, 2010

We have two assets, asset A and asset B, and the market portfolio, asset M. The return on these assets is comprised of an expected return (deterministic) and an unexpected return (random). We will assume that Z is a normally-distributed random variate with mean zero and variance one. We will model asset returns as a Brownian motion such that the equation for assets returns are...

$$r_m = (\mu_m - \frac{1}{2}\sigma_m^2)t + \sigma_m \sqrt{t} Z_m$$
$$r_a = (\mu_a - \frac{1}{2}\sigma_a^2)t + \sigma_a \sqrt{t} Z_a$$
$$r_b = (\mu_b - \frac{1}{2}\sigma_b^2)t + \sigma_b \sqrt{t} Z_b \tag{1}$$

We want to model the returns for asset A and B as depending in part on the market return. Another way of saying this is that the returns on asset A and B depends on the market return (systematic) and an idosyncratic return (unsystematic or company specific). We add correlation to the models for asset returns by correlating the random variables (the Z's in this case). We do this via a Gaussian Copula. The random variable Z_m is purely random. The Greek symbol rho will represent the correlation of the asset return with the market return. The random variables Z_a and Z_b are correlated with Z_m and here is how that is done...

$$Z_a = \rho_{am}\, Z_m + \sqrt{1 - \rho_{am}^2}\, \epsilon_a \tag{2}$$
$$Z_b = \rho_{bm}\, Z_m + \sqrt{1 - \rho_{bm}^2}\, \epsilon_b \tag{3}$$

The Greek symbol epsilon represent normally-distributed random variates with mean zero and variance one **that are independent**, i.e. not correlated.

We will assume that the correlation of asset A with the market equals the correlation of asset B with the market such that we can remove the subscripts assigned to the Greek symbol rho. The above equations become...

$$Z_a = \rho\, Z_m + \sqrt{1 - \rho^2}\, \epsilon_a \tag{4}$$
$$Z_b = \rho\, Z_m + \sqrt{1 - \rho^2}\, \epsilon_b \tag{5}$$

The first moment of the distribution of Z_a is...

$$\begin{aligned}\mathbb{E}\Big[Z_a\Big] &= \mathbb{E}\Big[\rho\, Z_m + \sqrt{1-\rho^2}\,\epsilon_a\Big] \\ &= \mathbb{E}\Big[\rho\, Z_m\Big] + \mathbb{E}\Big[\sqrt{1-\rho^2}\,\epsilon_a\Big] \\ &= \rho\,\mathbb{E}\Big[Z_m\Big] + \sqrt{1-\rho^2}\,\mathbb{E}\Big[\epsilon_a\Big] \\ &= 0\end{aligned} \tag{6}$$

...because $\mathbb{E}\Big[Z_m\Big]$ and $\mathbb{E}\Big[\epsilon_a\Big]$ are both equal zero by definition.

The second moment of the distribution of Z_a is...

$$\begin{aligned}\mathbb{E}\Big[(Z_a)^2\Big] &= \mathbb{E}\Big[\Big(\rho\, Z_m + \sqrt{1-\rho^2}\,\epsilon_a\Big)^2\Big] \\ &= \mathbb{E}\Big[\rho^2 Z_m^2 + 2\rho\sqrt{1-\rho^2}\,Z_m\epsilon_a + (1-\rho^2)\epsilon_a^2\Big] \\ &= \mathbb{E}\Big[\rho^2 Z_m^2\Big] + \mathbb{E}\Big[2\rho\sqrt{1-\rho^2}\,Z_m\epsilon_a\Big] + \mathbb{E}\Big[(1-\rho^2)\epsilon_a^2\Big] \\ &= \rho^2\mathbb{E}\Big[Z_m^2\Big] + 2\rho\sqrt{1-\rho^2}\,\mathbb{E}\Big[Z_m\epsilon_a\Big] + (1-\rho^2)\mathbb{E}\Big[\epsilon_a^2\Big] \\ &= \rho^2 + 1 - \rho^2 \\ &= 1 \end{aligned} \tag{7}$$

...because $\mathbb{E}\Big[Z_m^2\Big]$ and $\mathbb{E}\Big[\epsilon_a^2\Big]$ are both equal one by definition and $\mathbb{E}\Big[Z_m\epsilon_a\Big]$ is equal to zero by definition.

The expectation of the product of Z_a and Z_b is...

$$\begin{aligned}\mathbb{E}\Big[Z_aZ_b\Big] &= \mathbb{E}\Big[\Big(\rho\, Z_m + \sqrt{1-\rho^2}\,\epsilon_a\Big)\Big(\rho\, Z_m + \sqrt{1-\rho^2}\,\epsilon_b\Big)\Big] \\ &= \mathbb{E}\Big[\rho^2 Z_m^2 + \rho\sqrt{1-\rho^2}Z_m\epsilon_a + \rho\sqrt{1-\rho^2}Z_m\epsilon_b + (1-\rho^2)\epsilon_a\epsilon_b\Big] \\ &= \mathbb{E}\Big[\rho^2 Z_m^2\Big] + \mathbb{E}\Big[\rho\sqrt{1-\rho^2}Z_m\epsilon_a\Big] + \mathbb{E}\Big[\rho\sqrt{1-\rho^2}Z_m\epsilon_b\Big] + \mathbb{E}\Big[(1-\rho^2)\epsilon_a\epsilon_b\Big] \\ &= \rho^2\mathbb{E}\Big[Z_m^2\Big] + \rho\sqrt{1-\rho^2}\mathbb{E}\Big[Z_m\epsilon_a\Big] + \rho\sqrt{1-\rho^2}\mathbb{E}\Big[Z_m\epsilon_b\Big] + (1-\rho^2)\mathbb{E}\Big[\epsilon_a\epsilon_b\Big] \\ &= \rho^2 \end{aligned} \tag{8}$$

...because $\mathbb{E}\Big[Z_m^2\Big]$ is equal one by definition and $\mathbb{E}\Big[Z_m\epsilon_a\Big]$ and $\mathbb{E}\Big[Z_m\epsilon_b\Big]$ and $\mathbb{E}\Big[\epsilon_a\epsilon_b\Big]$ are equal to zero by definition.

Putting It All Together

$$\mathbb{E}\Big[Z_a\Big] = \mathbb{E}\Big[Z_b\Big] = 0 \tag{9}$$

$$\mathbb{E}\Big[Z_a^2\Big] = \mathbb{E}\Big[Z_b^2\Big] = 1 \tag{10}$$

$$\mathbb{E}\Big[Z_aZ_b\Big] = \rho^2 \tag{11}$$

We need to test the mean of Z_a, and also Z_b, which is...

$$\begin{aligned} mean_a &= \mathbb{E}\Big[Z_a\Big] \\ mean_a &= mean_b = 0 \end{aligned} \tag{12}$$

This is what we want because the means of the normally-distributed random variates are zero by definition.

We first need to calculate the variance of Z_a, and also Z_b, which is...

$$\begin{aligned} var_a &= \mathbb{E}\Big[Z_a^2\Big] - \Big[\mathbb{E}\Big[Z_a\Big]\Big]^2 \\ var_a &= 1 - 0^2 \\ var_a &= var_b = 1 \end{aligned} \tag{13}$$

This is what we want because the variances of the normally-distributed random variates are one by definition.

We want to determine ***pairwise*** correlation between Z_a and Z_b which is...

$$\begin{aligned}\rho_{ab} &= \frac{cov_{ab}}{stdev_a \, stdev_b} \\ &= \frac{\mathbb{E}\Big[Z_a Z_b\Big] - \mathbb{E}\Big[Z_a\Big]\mathbb{E}\Big[Z_b\Big]}{\sqrt{var_a}\sqrt{var_b}} \\ &= \frac{\rho^2 - 0 \times 0}{1 \times 1} \\ &= \rho^2 \end{aligned} \tag{14}$$

Conclusion...

Given the assumptions above the pairwise correlation between asset A and asset B is the square of the correlation of Asset A with the market or asset B with the market (because they are equal by assumption).